On The Foundation of Performance Measures under Asymmetric Returns

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Abstract. We examine two performance measures advocated for asymmetric return distributions: the Sortino Ratio - originally introduced by Sortino and Price (1994) - and a measure based on power utility introduced in Leland (1999). In particular, we investigate the role of the Maximum Principle in this context, and assess the conditions under which the measures satisfy it. Our results add further motivation for the use of a modified Sortino Ratio, by placing it on a sound theoretical foundation. In this light, we discuss its relative merits compared with alternative approaches.

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1. Introduction

When distributions are symmetric and the classical mean-variance Capital Asset Pricing Model (CAPM) - most often attributed to Sharpe (1964), Lintner (1965) and Mossin (1969) - valid, performance measures can be extracted directly from the model. In particular, from the empirical representation, three classical statistics are most commonly employed: the Sharpe Ratio, introduced in Sharpe (1966); the Treynor Index, derived in Treynor (1965); and Jensen's Alpha, which was developed by Jensen (1972). However, when returns are asymmetric and mean-variance rules no longer

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efficient, these measures cease to capture the essential features of the distribution. Whilst this has been a widely recognised problem in Finance, not least due to the presence of skewness in much financial data, only recently have remedies for this been advanced.

Sortino and Van der Meer (1991) advance the square root of the second lower partial moment, i.e.

$$\theta_{r_p}(t) = \left(\int_{-\infty}^{t} (t - \widetilde{r}_p)^2 \, pdf\,(\widetilde{r}_p)\, d\widetilde{r}_p \right)^{\frac{1}{2}} \tag{1}$$

as a measure of risk, rather than standard deviation. In (1), t can be a fixed, observable or random target, or a combination of these categories. It is often defined as the rate of return on a long-dated bond or Treasury Bill. Also, $\widetilde{r}_p$ denotes return on portfolio p with density function $pdf\,(\widetilde{r}_p)$. We shall follow the popular literature and define the above as the *semi-standard deviation* and its square as the *semi-variance*, although strictly speaking such terminology is correct only when $t = E(\widetilde{r}_p)$. The semi-variance only takes positive values for returns below t and is therefore sensitive to both skewness in the data and the probability of shortfalls, unlike variance, which weighs extreme positive and negative outcomes equally. Based on this measure of risk, Sortino and Price (1994) introduce the Sortino Ratio,

$$\frac{\overline{r}_p - t}{\theta_{r_p}(t)} \tag{2}$$

where θ_{r_p} is the semi-standard deviation and $\overline{r}_p$ denotes average returns. A modified version of this, namely

$$\frac{\overline{r}_p - r}{\theta_{r_p}(t)} \tag{3}$$

where r denotes the risk-free asset, is put forward as a competing performance measure in this article. This is equivalent to the Sharpe Ratio except standard deviation has been replaced by the semi-standard deviation (1) in the denominator[1]. We note that, although the earlier performance measure (2) implies that the riskless asset would have infinite performance when $t \leq r$, this is not the case with (3), when

[1] We have to exclude the case $t > r$, since this implies that, at $\overline{r}_p = r, \theta_{r_p}(t) \neq 0$, and a riskless asset is not regarded as riskless.

the performance of the riskless asset becomes undefined (as it does with the Sharpe Ratio). In this article, we shall add support for the use of (1) as risk measure by drawing on theoretical risk work and equilibrium arguments, essentially showing it relies upon assumptions no more obscure than the Sharpe Ratio itself.

Very recently, Leland (1999) has used options strategies and equilibrium theory to illustrate the shortcomings of CAPM-based performance measures, and argued for alternative approaches. In particular, he uses results from Rubinstein (1976), Brennan (1979) and He and Leland (1993) to argue the case for a representative agent "CAPM" based on power utility. The power utility model implies the empirical equation

$$E[\widetilde{r}_p - r] = \left(\frac{cov\left[\widetilde{r}_p, (1+\widetilde{r}_m)^{-b}\right]}{cov\left[\widetilde{r}_m, (1+\widetilde{r}_m)^{-b}\right]} \right) E[\widetilde{r}_m - r] \tag{4}$$

where b is a constant such that $b > 0$, $\widetilde{r}_m$ is the return on the market and r is the risk-free rate of return. This model can capture skewness in data, since expanding expected utility gives an expression in the higher moments of returns. Also, Leland shows that under certain market conditions, the risk measure

$$B_p = \frac{cov\left[\widetilde{r}_p, (1+\widetilde{r}_m)^{-b}\right]}{cov\left[\widetilde{r}_m, (1+\widetilde{r}_m)^{-b}\right]} \tag{5}$$

determines the risk of any fairly priced asset, even those exhibiting highly asymmetric returns. The performance measure Leland favours is the analogue of Jensen's Alpha (see Jensen (1972) for details) in this setting, namely

$$A_p = E[\widetilde{r}_p] - r - B_p(E[\widetilde{r}_m - r]) \tag{6}$$

This measures excess return after correcting for risk using a general equilibrium risk measure derived from a representative agent with power utility.

Although both Sortino's and Leland's work are motivated by the need for skewness to be captured by equilibrium risk measures, there might appear to be sharp differences between their approaches, not least because Leland's aim is to look at excess performance in a fashion similar to Jensen's Alpha, whilst the Sortino Ratio extends the Sharpe Ratio. It is our contention that one can motivate both these classical performance measures by similar arguments, and we shall use such methods to provide support for the original and modified Sortino Ratios, which hitherto have not been formally placed in a theoretical framework. This thus also addresses an assertion by Leland, who refers to Sortino and Price's extension (2) as an "*ad hoc*" measure which is "*not grounded in capital market equilibrium theory*" (see Leland (1999), page 34, endnote 4), and will hopefully help dispel possible doubts amongst practitioners considering its use.

The paper is organised as follows: In the next section, we present the Maximum Principle, which is commonly deemed a necessary property of a good performance measure, and discuss its relevance for the modified Sortino Ratio (3): in particular, Section 2.1 looks at the nature of the mean-semi standard deviation efficient frontier, whilst Section 2.2 considers the possibility of deriving a representative agent with mean-semi standard deviation preferences using support from both risk theory and economic equilibrium arguments. Section 3 is reserved for our conclusions.

2. The Maximum Principle and the modified Sortino Ratio

We shall put forward a view of a performance measure as a measure that ranks portfolios both for individual investors and for the representative investor, who holds the market portfolio. Such a measure should be maximised by the representative investor, so that if one exceeds that maximum, the presence of special information or skills can be inferred. This maximality of the market is what we refer to as the Maximum Principle. The idea behind this already exists in the literature. Most notably, Grinblatt and Titman (1989) propose a measure referred to as the "positive period measure", which is zero for the market, but positive in the presence of good market timing and/or stock selection abilities. The next Proposition presents two conditions under which a performance measure in general satisfies the Maximum Principle.

Proposition 1. The Maximum Principle is satisfied if either of the following two conditions is satisfied:

1. The performance measure is based on the maximised expected utility of the representative agent.

2. The performance measure is the ratio of the excess expected return to the risk

measure, the efficient frontier is linear and the market portfolio is efficient.

Proof. In case 1, the proof is immediate. In case 2, since the efficient frontier is linear, the ratio of expected excess return to the risk measure along it is constant. Thus, since the market portfolio is efficient and lies on the frontier, it maximises the performance measure ■

Whilst there may be other conditions that guarantee the Maximum Principle, Proposition 1 implies that Leland's performance measure (6) satisfies the Maximum Principle by the first condition, whilst the Sharpe Ratio satisfies the Maximum Principle by the second condition. The question that now arises, is whether the modified Sortino Ratio (3) will satisfy the Maximum Principle. In accordance with Proposition 1, such an analysis can be approached in at least two ways. We shall first investigate the nature of the mean-semi standard deviation efficient frontier; in Section 2.2, we then examine the conditions in which one gets a representative agent with mean-standard deviation decision characteristics.

2.1. The mean-semi standard deviation frontier. In this section, we examine the nature of the efficient frontier when investors have mean-semi standard deviation preferences, where we make the assumption that Two Fund Money Separation (TFMS) - i.e. that all investors choose a mixture of the riskfree asset and the same risky portfolio, though not necessarily the same mixture - is attained. We let μ_p denote the mean return of a portfolio p with return $\widetilde{r}_p$, w is the weight in the risky portfolio with return $\widetilde{r}_{k,}$ and r is the riskfree return. Then, by definition, for w assumed positive,

$$\widetilde{r}_p = w\widetilde{r}_k + (1-w)r \tag{7}$$

Furthermore, if the target, t, is a given constant, the semi-variance can be written as

$$\theta^2_{r_p}(t) = \int_{-\infty}^{t} (t-\widetilde{r}_p)^2 \, pdf\left(\widetilde{r}_p\right) d\widetilde{r}_p \tag{8}$$

where $pdf\left(\widetilde{r}_p\right)$ denotes the probability density function of $\widetilde{r}_p$. To illustrate our inferences about the $\left(\mu_p, \theta_{r_p}\right)$ - frontier, we find working with $pdf\left(\widetilde{r}_k\right)$ more informative.

By changing variables, we get

$$\theta_{r_p}^2(t) = \int_{-\infty}^{\frac{1}{w}(t-(1-w)r)} (w(\widetilde{r}_k - r) + r - t)^2 pdf\left(\widetilde{r}_k\right) d\widetilde{r}_k \tag{9}$$

By differentiating (9) w.r.t. w, substituting for $\frac{\partial \mu_p}{\partial w} = \mu_k - r$, and for w by rearranging the expectation of (7) - i.e. $w = \frac{\mu_p - r}{\mu_k - r}$ - we derive the $\left(\mu_p, \theta_{r_p}\right)$ - frontier given by

$$\theta_{r_p}(t) = \frac{(\mu_p - r)^{\frac{1}{2}}}{\mu_k - r} \left[\int_{-\infty}^{\frac{(\mu_k - r)t - (\mu_k - \mu_{p)r}}{\mu_p - r}} \left[\left(\mu_p - r\right)\left(\widetilde{r}_k - r\right) + (r - t)(\mu_k - r)\right]^2 pdf(\widetilde{r}_k) d\widetilde{r}_k \right]^{\frac{1}{2}} \tag{10}$$

This is clearly non-linear, which has implications for the concavity/convexity of the efficient frontier, and thus also for equilibrium arguments and the Maximum Principle. Harlow and Rao (1989) show that the frontier of $\left(\mu_p, \theta_{r_p}^2\right)$ is concave - however, this has no implication for the concavity/convexity of $\left(\mu_p, \theta_{r_p}\right)$. The following theorem resolves this issue; the proof is in the Appendix.

Theorem 1. In the TFMS case, for $t < r$, the $\left(\mu_p, \theta_{r_p}\right)$ - frontier is concave if $0 < w < 1$ and convex for $w > 1$. At $w = 1$ it has a point of inflection.. If $t = r$, the frontier is linear.

It follows from the above theorem that when $t < r$, in order to maximise the Sortino Ratio one should increase w when $w > 1$, and decrease w when $w < 1$. Hence, if short sales (and borrowing) are disallowed, the Sortino Ratio is maximised at the point where one holds zero equity. A representative agent who follows the Maximum Principle would thus hold only bonds, which is not consistent with an equilibrium in any economy with a positive supply of stocks. Indeed, these are the arguments that have raised doubts about the Sortino Ratio. The question then arises as to whether there are other conditions in which the Maximum Principle may be true. The following Proposition establishes such a condition.

Theorem 2. If $w > 0$, so the representative agent holds some equity, and the representative agent has a mean-semivariance utility function - to be defined in (13)

- with target

$$t = r + wt^* \tag{11}$$

for some $t^* \leqslant 0$, then the mean-semi standard deviation frontier is linear and the modified Sortino Ratio (3) satisfies the Maximum Principle.

Proof. Under this parameterisation, by substituting (11) into (9), one observes that $\theta^2_{r_p}(t)$ simplifies to

$$\begin{aligned} \theta^2_{r_p}(t) &= w^2 \int_{-\infty}^{r+t^*} \left[(\widetilde{r}_k - r) - t^*\right]^2 pdf(\widetilde{r}_p) d\widetilde{r}_p \\ &= w^2 \int_{-\infty}^{r+t^*} \left[\widetilde{r}_k - (r + t^*)\right]^2 pdf(\widetilde{r}_p) d\widetilde{r}_p \\ &= w^2 \theta^2_{r_p}(r + t^*) \end{aligned} \tag{12}$$

Hence, from (10) and (12), it is apparent that both θ_{r_p} and $\mu_p - r$ are linear in w; thus, the $\left(\mu_p, \theta_{r_p}\right)$- frontier is linear ■

We have thus established conditions in which the modified Sortino Ration satisfied the Maximum Principle. One feature of this result is that the condition (11) would imply targets changing with the optimal allocation w; i.e. the target would be endogenous and so the axioms of expected utility may not hold. (In the conventional case, the target would have been exogenous to the process.) Without digressing into a discussion of the merits or failings of Expected Utility Theory, we mention that there is a large literature which deals with examining how target-based preference may be applied in Finance without necessarily satisfying the expected utility axioms (e.g. see Francke and Weber (1997) or Ang, Bekeart et al (2000) where a large bibliography can be found) precisely by making targets endogenous or functions of initial wealth. This is furthermore related to the standard practice that a portfolio composed of different asset classes (e.g. cash, bond, equity), would need different benchmarks for each class.

Thus, there are plausible criteria under which the Sortino Ratio might satisfy the Maximum Principle. We next explore this further by considering a utility representation of the mean-semi standard deviation preferences and examine when a representative agent exists in such conditions.

2.2. Representative Agent and Equilibrium. The Leland (1999) approach to performance measurement involves showing the existence of a representative agent (in his case with power utility) whose optimal portfolio is that of the market as a whole. In this section, we shall investigate the representative agent in the case where individual investors have mean-semi standard deviation preference, discussing the assumptions necessary to ensure existence. We further include some motivation on the use of mean-standard deviation preferences to address Leland's criticisms of the Sortino Ratio, as outlined in the Introduction.

The link between mean-semi standard deviation preferences and expected utility was formally established by Fishburn (1977), who showed that a decision rule based on expected returns and semi-variance (and thus also semi-standard deviation (1)) is congruent with Expected Utility Theory only if the utility function[2] takes the form

$$U(\widetilde{r}_p) = \begin{cases} \widetilde{r}_p & \widetilde{r}_p \geq t \\ \widetilde{r}_p - c\,(t - \widetilde{r}_p)^2 & \widetilde{r}_p < t \end{cases} \tag{13}$$

where t and c are constants, where c satisfies $c > 0$. Note that by taking expectations of (13), one gets

$$\begin{aligned} E\left[U(\widetilde{r}_p)\right] &= E(\widetilde{r}_p) - cE\left[(t - \widetilde{r}_p)^2(\widetilde{r}_p < t)\right] \\ &= E(\widetilde{r}_p) - c\int_{-\infty}^{t} (t - \widetilde{r}_p)^2 pdf(\widetilde{r}_p)d\widetilde{r}_p \\ &= \mu_p - c\theta^2_{r_p}(t) \end{aligned} \tag{14}$$

Thus, expected utility is a trade-off between mean and semi-variance. Essentially, this is analogous to quadratic utility in conventional mean-variance analysis except the risk measure is now (1) rather than conventional standard deviation. This can be contrasted with the Leland (1999) utility function which, despite following from his assumptions, does not explicitly transform to comparable risk and return measures.

[2] In fact, Fishburn's result holds for (13) where the exponent of $(t - \widetilde{r}_p)$ is $b \geq 2$. Hence, there are other risk measures for which the congruency can be established. For the purposes of this paper, we restrict ourselves to the relevant case $b = 2$.

The choice of (1) as risk measure and consequently (13) as utility function is far from random. In Fishburn (1980 and 1981), the author gives an elegant justification for the use of formal asymmetric risk measures including the semi-standard deviation (1), by imposing a series of plausible restrictions on the underlying preference relations. His general approach was to split the return distribution into a loss probability, a gain probability and conditional measures of return levels *given* a loss or gain, where loss/gain was measured relative to an arbitrary target, t. Such a structure has since been adapted by financial econometricians (see for instance Knight et al. (1995)) as an appropriate way to statistically model asymmetric financial time series. More recently, the use of the Sortino Ratio as a performance measure in practice has also been evident by its application on a large number of Web-based performance rating companies (see, for instance http://www.cta-online.com/, http://www.cradv.com/ or http://www.worthtrading.com/).

The use of downside risk measures has been advocated by numerous academics and practitioners in Finance, and serve a broad range of functions. As early as the original mean-variance text of Markowitz (1952), the author points to the semi-variance as an attractive risk measure, but is dissuaded from examining it further at the time due to it's computational and algebraic unfriendliness. Later authors have linked downside risk to stochastic dominance and popular risk-ranking literature (see, for instance, Bawa (1975) or Menezes, Geiss et al. (1980)). Several other advance asset pricing models based on numerous different downside risk measures (see Roy (1952), Bawa and Lindenberg (1977), Harlow and Rao (1989) or Salomon Brothers (1989)). Empirically, these models were shown to be favourable to CAPM at explaining small company returns (see Pedersen (1998)). For an extensive bibliography of relevant works, we refer the reader to Pedersen (1999b). Very recently downside risk was taken in a new direction and linked with derivatives portfolio optimisation (see Huang et al. (2001) and Pedersen (2001)). Finally,we mention that the large amount of interest in loss aversion utility functions, which can be extracted from Prospect Theory (see Kahnemann and Tversky (1979)) and were illustrated in Fishburn and Kochenberger (1979), is yet another reason why semi-variance and other downside risk measures have received renewed interest. We hence claim that there is nothing *ad hoc* at all about the choice of risk measure in the denominator of the Sortino Ratio (2). However, the question of whether this, or the modified Sortino Ratio (3), can be justified as a performance measure in an equilibrium setting, still needs to be addressed.

In this regard, we shall compare the underlying assumptions needed to establish the required conditions with the standard assumptions made in traditional and comparable models. The Leland (1999) performance measure (6) and the three classical performance measures (the Sharpe Ratio, Treynor's Index and Jensen's Alpha) are all rooted in versions of the Capital Asset Pricing Model, originally presented by Sharpe (1964), Lintner (1965) and Mossin (1969) but where, rather than quadratic utility, Leland uses a representative agent who has power utility with an exponent less than one. Bawa and Lindenberg (1977) and Harlow and Rao (1989) derived equilibrium pricing models which were identical to CAPM, except investors minimise the semi-standard deviation (1). However, as was pointed out by Chow and Denning (1994), neither set of authors proves the existence of equilibrium without making assumptions that automatically validate the original mean-variance CAPM. Hence, they argued, there were still no conditions presented in the literature proving the existence of such downside risk asset-pricing models in a world where CAPM was not equally suitable. The Sharpe Ratio remained the appropriate, best theoretically supported, performance measure.

We will draw upon a result in Satchell (1996) - which was later generalised in Pedersen (1999a) to include more general asymmetric preferences - and consider an equilibrium model obtained under certain parameter restrictions, but without restrictive distributional assumptions. To do this, we assume that individual investors have piece-wise utility functions as given by (13), but otherwise maintain all the standard assumptions of CAPM. The crucial step is to prove that two fund money separation (TFMS) - i.e. the property that all investors choose to invest in a mixture of the riskless asset and a fixed risky portfolio - obtains, which enables aggregation and thus allows one to characterise the utility function of the representative agent. We thus state two theorems, which are the main results in Satchell (1996); their proofs, and further details and discussion, can be found in the original text. The first theorem addresses TFMS, making trivial non-restrictive distributional assumptions.

Theorem 3. (Satchell (1996)): Suppose that there are K individuals indexed $k = 1, 2, ...K$. Individual k has positive initial wealth W_{0k}, wealth target $h^k = W_{0k}(1 + t^k)$, where $t^k < r$, and utility function

$$U_k(\widetilde{W}) = \begin{cases} \widetilde{W} & \widetilde{W} \geq h^k \\ \widetilde{W} - \lambda^k(h^k - \widetilde{W})^2 & \widetilde{W} < h^k \end{cases} \tag{15}$$

where $\lambda^k > 0$ and h^k is a real number. In addition, we assume that

$$\lambda^k = \left(W_{0k} f^k\right)^{-2} \tag{16}$$

for all k, where

$$f^k = 1 - \frac{t^k}{r} > 0 \tag{17}$$

and that the joint distribution of returns assigns positive probability to the events $\widetilde{W} < h^k$ and $\widetilde{W} \geq h^k$. Then, if all investors hold the risky portfolio long, TFMS obtains and f^k is the fraction invested by agent k in the risky portfolio.

We reiterate that expected utility given (15) was shown earlier to be of the form "expected return minus risk", where risk is denied as (1), as discussed earlier. This result hence gives conditions under which the Maximum Principle may be satisfied for the modified Sortino Ratio through the second part of Proposition 1. The following theorem establishes the existence of a representative agent in an economy full of agents with such preferences.

Theorem 4. (Satchell (1996)): When all individuals satisfy the conditions of the last theorem, aggregate demand is identical with that of a single representative consumer whose initial wealth is $W_m = \sum_k W_{0k}$, target $h^m = W_m(1 + t^m) = \sum_k h^k$, where $t^m = \frac{\sum_k W_{0k} t^k}{W_m} < r$, and utility function given by

$$U(\widetilde{W}) = \begin{cases} \widetilde{W} & \widetilde{W} \geq h^m \\ \widetilde{W} - \lambda(h^m - \widetilde{W})^2 & \widetilde{W} < h^m \end{cases} \tag{18}$$

where $\lambda = (W_m f_m)^{-2}$ and $f_m = 1 - \frac{t^m}{r} > 0$.

Hence, there are conditions under which TMFS and aggregation holds for an economy, in which people have mean-downside risk preferences. We note that the above results are not in conflict with the necessity results on TFMS in Cass and Stiglitz (1970) since our wealth targets, h^k, which are part of the specification of investor's utility functions, depend on initial wealth W_{0k}, a similar restriction to that of having optimal allocations affecting targets, as in Theorem 2. Having showed the existence of a representative agent, we state the following corollary without proof, which describes the representative agent in the case of a linear frontier, which thus formally links the analysis with that conducted in the previous section.

Corollary. If $t^m = r + f_m t_m^* < 0$, as in Theorem 2, the $\left(\mu_p, \theta_{r_p}\right)$ - frontier is linear so that the modified Sortino Ratio (3) satisfies the Maximum Principle.

This argument can be taken one step further. When a representative agent with utility function $U(\widetilde{W})$ exists, the first order conditions of his optimisation problem can be rearranged (see, for instance, Huang and Litzenberger (1988), Chapter 6) to give

$$E\left[\widetilde{r}_p - r\right] = \left(\frac{cov\left[\widetilde{r}_p, U'(\widetilde{W})\right]}{cov\left[\widetilde{r}_m, U'(\widetilde{W})\right]} \right) E\left[\widetilde{r}_m - r\right] \tag{19}$$

This relationship holds for all utility functions which are strictly increasing and concave. With power utility, this yields the Leland model (4). For our representative agent's utility function (18), which satisfies the sufficient conditions to apply (19), one gets the equilibrium risk measure

$$\beta = \frac{cov\left[\widetilde{r}_p, \Delta(t - r_m)^2\right]}{cov\left[\widetilde{r}_m, \Delta(t - r_m)^2\right]} \tag{20}$$

where $\Delta = 1$ if $\widetilde{W} < h$ and 0 otherwise. Hence, purely by choosing alternative utility functions and making no market assumptions other than those in the traditional CAPM, an alternative capital markets equilibrium characterisation may be established. This result should be compared with the results of Bawa and Lindenberg (1977) and Harlow and Rao (1989), where similar beta's were derived, albeit without a representative agent.

The results we have demonstrated in this section were derived on the grounds of Proposition 1(1), namely that performance is based on the maximised expected utility of the representative agent. In doing so, we have also demonstrated Proposition 1(2), i.e. that the frontier can be linear. Together, these arguments show that the models which lead to the use of more general performance measures such as the Sortino Ratio are based on theoretical foundations comparable to those underpinning the Sharpe Ratio and, what is more, use similar representative agent and equilibrium techniques to those applied in the power-utility CAPM advocated by Leland (1999).

3. Conclusions

A sensible measure of an economic or financial relationship customarily follows as the consequence of an equilibrium in a model that is consistent with known characteristics of observable data. Our arguments for the use of the semi-variance are to correct the belief that the semi-variance has no theoretical foundations and to advance the modified Sortino Ratio a as performance measure, whose origins are as sound as alternative measures. To support this, we have referred to existing axiomatic approaches to downside and asymmetric risk from Mathematical Psychology, which derive the semi-variance from first principles, and have listed further advocates of the use of downside risk. Also, we have shown that there exists a utility-based one-period capital market equilibrium model which is akin to replacing the standard deviation by semi-standard deviation as risk measure in CAPM, and thus provides motivation for the Sortino Ratio exactly like the CAPM promotes use of the Sharpe Ratio. This model does not rest on assumptions about the distributional properties of the market portfolio, but requires the existence of a representative agent; we have given results which prove that, under certain conditions, such a representative agent exists.

Practitioners today already use semi-variance as a risk measure and mean-semi variance optimization methods have been used sporadically since their introduction by Markowitz (1952). We hope our results go some way to remove potential doubt about the appropriateness of using measures derived from downside risk considerations (and the Sortino Ratio in particular) amongst practitioners, and encouraged further development in this field.

Proof of Theorem 1

The case for $t = r$ is proved in Harlow and Rao (1989). Recalling (7)

$$\widetilde{r}_p = w\widetilde{r}_k + (1-w)r \tag{21}$$

for $t < r$, it suffices to show that, when $0 < w < 1$,

$$(1-w)\theta_r(t) + w\theta_{r_k}(t) \leq \theta_{r_p}(t) \tag{22}$$

and when $w > 1$,

$$(1-w)\theta_r(t) + w\theta_{r_k}(t) \geq \theta_{r_p}(t) \tag{23}$$

Now, since $t < r$, $\theta_r(t) = 0$, it suffices to show that

$$w\theta_{r_k}(t) \geq \theta_{r_p}(t) \Leftrightarrow w \geq 1 \tag{24}$$

To see this is true, note that $w\theta_{r_k}(t) \geq \theta_{r_p}(t) \Leftrightarrow w^2\theta^2_{r_k}(t) \geq \theta^2_{r_p}(t)$. Hence, using (9),

$$\begin{aligned} \theta^2_{r_p}(t) &= w^2 \int_{-\infty}^{\frac{1}{w}(t-(1-w)r)} \left(\widetilde{r}_k - \left[\frac{t-(1-w)r}{w}\right]\right)^2 pdf\left(\widetilde{r}_k\right) d\widetilde{r}_k \\ &= w^2\theta^2_{r_k}\left(\frac{t-(1-w)r}{w}\right) \end{aligned} \tag{25}$$

Now, by definition, $\theta^2_{r_f}(t)$ is an increasing function of t (since, if the target increases, a larger part of the distribution contributes to risk). Thus,

$$\begin{aligned} \theta^2_{r_p}(t) &= w^2\theta^2_{r_k}\left(\frac{t-(1-w)r}{w}\right) \geq w^2\theta^2_{r_k}(t) \\ &\Leftrightarrow \frac{t-(1-w)r}{w} > t \\ &\Leftrightarrow t(1-w) > r(1-w) \end{aligned} \tag{26}$$

Now, since $t < r$, $t(1-w) > r(1-w) \Leftrightarrow w > 1$ and the result holds ■

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